2 Distributions

Exercise 2.1. Let $T \in \mathscr{E}'(\mathbb{R}^d)$ and $\varphi \in \mathscr{D}(\mathbb{R}^d)$. Show that $T * \varphi$ is an element of $\mathscr{D}(\mathbb{R}^d)$ and $\mathrm{supp}(T * \varphi)$ is contained in $\mathrm{supp}\, T + \mathrm{supp}\, \varphi$.

Exercise 2.2. Let $T, S \in \mathscr{D}'(\mathbb{R}^d)$ be such that $T * \varphi = S * \varphi$ for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$. Show that $T = S$.

Exercise 2.3. Use Exercise 2.2 to show that
— $T * (\varphi * \psi) = (T * \varphi) * \psi$ for any choice of $T \in \mathscr{D}'(\mathbb{R}^d)$ and $\varphi, \psi \in \mathscr{D}(\mathbb{R}^d)$,
— $\delta * T = T * \delta = T$ for any $T \in \mathscr{D}'(\mathbb{R}^d)$.
Hint : recall that if $T \in \mathscr{D}'(\mathbb{R}^d)$ and $f \in \mathscr{D}(\mathbb{R}^d \times \mathbb{R}^d)$, then one has

$$\left\langle T, \int_{\mathbb{R}^d} f(\cdot, y)dy \right\rangle_{\mathscr{D}' \leftrightarrow \mathscr{D}} = \int_{\mathbb{R}^d} \langle T, f(\cdot, y)\rangle_{\mathscr{D}' \leftrightarrow \mathscr{D}}\, dy.$$

Exercise 2.4. Let $\Omega \subset \mathbb{R}^d$ be an open subset and $u \in \mathscr{D}'(\Omega)$ be such that $\Delta u = 0$. Show that u is a smooth function on Ω.

We recall the mean-value formula.

Theorem. *Let $\Omega \subset \mathbb{R}^d$, and $u \in C^\infty(\Omega)$. Then, u is harmonic ($\Delta u = 0$) on Ω, if and only if u satisfies the mean-value formula,* i.e. *for all $x \in \Omega$ and for all $0 < r < \mathrm{dist}(x, \partial\Omega)$,*

$$u(x) = \fint_{\partial B(x,r)} u(y)\, d\mathscr{H}^{d-1}(y) = \frac{1}{\beta(d) r^{d-1}} \int_{\partial B(x,r)} u(y)\, d\mathscr{H}^{d-1}(y),$$

where $\beta(d) = \mathscr{H}^{d-1}(S^{d-1})$ is the area of the unit sphere in $\mathbb{R}^d$.

Exercise 2.5. Let $u \in C^\infty(\Omega)$ be a harmonic function such that $u(x) \underset{|x|\to\infty}{\longrightarrow} 0$. Show that u is constant.

Exercise 2.6. Let $d \geq 2$. Show that for all compactly supported distribution $f \in \mathscr{E}'(\mathbb{R}^d)$, there exists infinitely many solutions $u \in \mathscr{D}'(\mathbb{R}^d)$ to the equation $\Delta u = f$. Show that all solutions are smooth outside of $\mathrm{supp}(f)$, and that the only solution u such that $u(x) \underset{|x|\to\infty}{\longrightarrow} 0$ is given by

$$u = G_d * f,$$

where G_d is the fundamental solution of the Laplace equation, given by

$$G_d(x) = \begin{cases} -\dfrac{1}{(d-2)\beta(d)} \dfrac{1}{|x|^{d-2}} & \text{for } d \geq 3 \\ \dfrac{1}{2\pi} \log|x| & \text{for } d = 2, \end{cases}$$